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Asymptotics of Peaks-over-Threshold Estimators in Long Memory Linear Time Series

joint work with M. Oesting & G. Stupfler

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Example:

With the **simple PoT** we count exceedances of (X_t) over u_n to estimate $\mathbb{P}[X_0 > u_n]$, i.e.,

$$\frac{1}{n}\sum_{t=1}^{n} \mathbb{1}\{X_t > u_n\} \qquad \text{to estimate} \qquad \mathbb{P}[X_0 > u_n]\,.$$

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Mostly used together with other estimators, e.g., moving window estimator

$$\frac{\frac{1}{n}\sum_{t=1}^{n} \mathbb{1}\{X_t > u_n, X_{t+h} > u_n\}}{\frac{1}{n}\sum_{t=1}^{n} \mathbb{1}\{X_t > u_n\}} \qquad \text{to estimate} \qquad \chi(h)\,,$$

where χ is the tail-dependence-coefficient

$$\chi(h) := \lim_{n\to\infty} \mathbb{P}[X_h > u_n \mid X_0 > u_n] = \lim_{n\to\infty} \frac{\mathbb{P}[X_h > u_n, X_0 > u_n]}{\mathbb{P}[X_0 > u_n]} \qquad \text{for } h > 0\,.$$

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Other Examples:

- **Hill Estimator** to estimate how heavy the tails are, that is,

$$\frac{1}{n}\sum_{t=1}^{n}\left(\log(X_t)\ -\ \log(u_n)\right)\mathbb{1}\{X_t > u_n\} \qquad \text{to estimate} \qquad \gamma > 0\,,$$

where $1/\gamma$ is the index of regular variation of the tail-function $x \mapsto \mathbb{P}[X_0 > x]$.

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- Empirical estimator for **Expected shortfall**:

$$\frac{\frac{1}{n}\sum_{t=1}^{n}(X_t \;-\; u_n)\,\mathbb{1}\{X_t > u_n\}}{\frac{1}{n}\sum_{t=1}^{n}\mathbb{1}\{X_t > u_n\}} \qquad \text{to estimate} \qquad \mathbb{E}[X_0 - u_n \mid X_0 > u_n]\,.$$

Outline

Asymptotics for i.i.d. Data

1

Asymptotics of PoT Estimators for i.i.d. Data

Theorem (CLT for i.i.d. data)

Let (X_n) be i.i.d. and (u_n) a sequence with $u_n \to \infty$ such that $n\mathbb{P}[X_0 > u_n] \to \infty$.

Takeaway:

What to expect from dependent data:

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- strong dependence (long memory) behavior different from i.i.d.

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Our Notion of Long Memory:

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Our Notion of Long Memory: (Causal) linear time series

$$X_t = \sum_{k=0}^{\infty} a_k \varepsilon_{t-k}, \qquad t \in \mathbb{N}_0$$

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- symmetric i.i.d. innovations (ε_k) with notion of heavy tails

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- slowly decaying coefficients (a_k) with $a_k \sim k^{-1+d}$ with long memory parameter

$$0 < d < 1 - 1/\alpha$$

  $\rightsquigarrow$ reminiscent of fractional differencing parameter in fractional ARIMA

Asymptotics for Long Memory Linear Time Series

Theorem (S., Oesting & Stupfler 2025+)

Let (X_t) be a long memory linear time series. Under technical conditions on the distribution of the innovations (ε_k) and the growth rate of (u_n) it holds

$$n^{1-d-1/\alpha}\frac{\mathbb{P}[X_0>u_n]}{f_{X_0}(u_n)}\frac{1}{n}\sum_{t=1}^{n}\left(\frac{\mathbb{1}\{X_t>u_n\}}{\mathbb{P}[X_0>u_n]}-1\right)\to_d C(\mathbb{P}_\varepsilon,d,\alpha)\cdot Z_\alpha \qquad \text{for } n\to\infty\,,$$

where

$$Z_\alpha \sim \begin{cases} S\alpha S\,, & \alpha\in(1,2)\,,\\ \mathcal{N}(0,1), & \alpha=2 \text{ and variance exists.}\end{cases}$$

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Remarkable:
The density of the time series f_{X_0} pops up in the ratio $\mathbb{P}[X_0>u_n]/f_{X_0}(u_n)$.

Idea of Proof

We extend ideas from (Koul & Surgailis 2001):

- Control growth of

$$\mathbb{E}\left[\left|\sum_{t=1}^{n} \mathbb{1}\{X_t > u_n\} - \mathbb{P}[X_0 > u_n] + f_{X_0}(u_n) X_t\right|^r\right]$$

  for some carefully chosen $r \in (1, \alpha)$.

- Use CLT for partial sums

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Also: We have proved more general results, e.g., for Hill estimator.

Ratio of Tail Function and Density

From i.i.d. section: expect to get punished for looking at extremes.
How severe punishment is we can read off:

$$\frac{\mathbb{P}[X_0 > u_n]}{f_{X_0}(u_n)} \sim \left\{ \right.$$

Observation:

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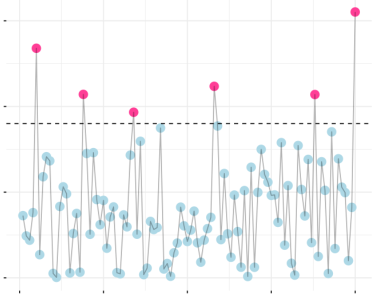

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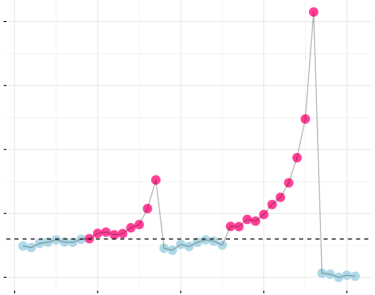

Random

Thresholds

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Tail Empirical Process

- Version with deterministic threshold

- Version with random threshold

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$$\widetilde{T}_n(s) \;=\; \frac{1}{n\mathbb{P}[X_0 > u_n]} \sum\nolimits_{t=1}^{n} \mathbb{1}\{X_t > u_n \cdot s\} \qquad \text{for } s > 0$$

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Theorem (Kulik & Soulier 2011)

In long memory stochastic volatility model:

- $\widetilde{T}$ shows non-standard behavior due to long memory
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Question: Do we find this in **long memory linear time series**?

Hill Estimator with Random Thresholds and Heavy Tails

We consider the Hill estimator with deterministic ($\widetilde{\gamma}_n$) and random ($\widehat{\gamma}_n$) thresholds.
(More direct proof by alternative tool.)

Theorem (S., Oesting & Stupfler 2025+)

With long memory linear time series, as $n \to \infty$, it holds

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\begin{aligned}
n^{1-(d+1/(\nu\wedge 2))} \quad u_n \quad (\widetilde{\gamma}_n - \mathbb{E}[\widetilde{\gamma}_n]) &\to_d \frac{\nu^2}{1+\nu} \quad Z_{\nu\wedge 2}\,, \\
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Stochastic Volatility Model

- no long memory for random thresholds

Linear Time Series

- phase transition between heavy/light tails ($\nu \to \infty$)

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Stochastic Volatility Model

- no long memory for random thresholds
- asymptotic independence $\chi(h) = 0$

Linear Time Series

- phase transition between heavy/light tails ($\nu \to \infty$)
- asymptotic dependence $\chi(h) > 0$

Light Tails

We consider a version of the "Hill estimator" with deterministic ($\widetilde{\gamma}_n$) and random ($\widehat{\gamma}_n$) thresholds.

Theorem (S., Oesting & Stupfler 2025+)

With long memory linear time series and second order condition as $n \to \infty$ it holds

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Deterministic Thresholds

- long memory effect in the rate $n^{1/2-d}$

Random Thresholds

- long memory rate **too slow**

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$$n^{1/2-d} \quad \frac{1}{q_{X_0}\left(1 - \frac{k}{n}\right)} \quad (\widehat{\gamma}_n - \mathbb{E}[\widehat{\gamma}_n]) \quad \to_{\mathbb{P}} \; 0 \,.$$

Deterministic Thresholds

- long memory effect in the rate $n^{1/2-d}$
- slower speed $1/u_n$ due to considering only extremes

Random Thresholds

- long memory rate **too slow**
- asymptotic profile unclear, but possibly **no long memory effect**

Summary

In Long Memory Linear Time Series...

- ... convergence rate of PoT estimators depends on heavy/light tails.
- ... with heavy tails we get rewarded for looking at extremes.
- ... transfer to random thresholds is beneficial both in practice and theory.

Pre-Print available:

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Thank You!

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