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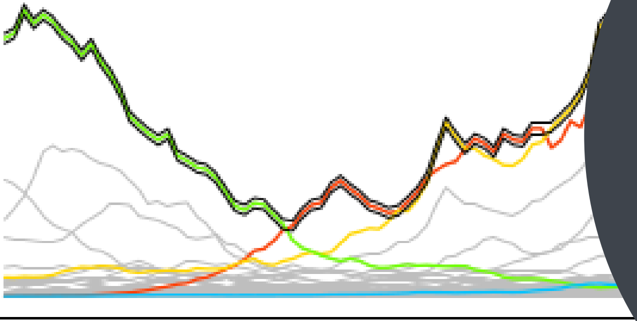

Extending Brown-Resnick stationarity to the max-domain of attraction

EcoSta2025

Outline

Motiva-tion

1

From a stochastic processes point of view

Setting: real-valued (time-homogeneous) Markov process $(X_t)_{t \geq 0}$

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Question: Can we construct stationary objects from σ-finite invariant measures?

From an extreme value theory point of view

Setting: (Z_t) is a max-stable process: for all $n \in \mathbb{N}$ there exist $a_n > 0, b_n \in \mathbb{N}$ such that

$$\max_{i=1,\dots,n} \frac{(Z_t)^{(i)} - b_n}{a_n} =_d (Z_t) \qquad \text{where } (Z_t)^{(i)} \text{ are i.i.d. copies of } (Z_t)$$

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- max-stable processes (with Gumbel margins) allow representation

$$Z_t = \max_{i \in \mathbb{N}} (U^{(i)} + X_t^{(i)}) \qquad \text{(de Haan, 1984)}$$

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- Brown-Resnick process (famous example):
  - $X_t = B_t - t/2$
  - Z_t is stationary even though X_t is not
    $\rightsquigarrow$ **Question:** When and why is (Z_t) stationary? **Answer:** (Kabluchko et. al., 2009)

Construction of stationary particle systems

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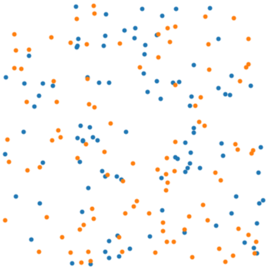
Ingredients

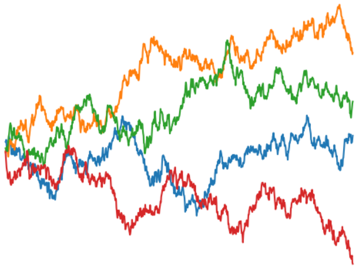
Point Processes

- Markovian particle systems
- stationarity via invariant measure



Stochastic Differential Equations

- SDE driven by Brownian motion
- invariant measures via Fokker-Planck equation



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Assumptions:

- (X_t) (time-homogeneous) Markov process
- $\{U^{(i)} \mid i \in \mathbb{N}\}$ points of a PPP with σ-finite Borel intensity measure ν

Theorem (Brown, 1970)

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$$\nu_{t_1,\ldots,t_k}[A] = \int_{\mathbb{R}} \mathbb{P}[(X_{t_1}, \ldots, X_{t_k}) \in A \mid X_0 = x] \, \mathrm{d}\nu(x)$$

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Takeaway: Invariant process-measure pair $((X_t), \nu)$ gives stationary particle system

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Next: SDE framework for constructing invariant process-measure pair

Invariant measure via SDE theory

Formalism:

$$\mathrm{d}X_t = d(X_t)\,\mathrm{d}X_t + \mathrm{d}B_t$$

Target:

- drift term d
- invariant measure $\eta = h\mathrm{d}x$

$\rightsquigarrow$ Related via stationary **Fokker-Planck equation**:

$$d(x)h(x) = \frac{h'(x)}{2}$$

Under additional regularity conditions—specifying either target—we get an invariant process-measure pair.

Example—from max-stable to max-id process

Question:
Can we allow for a different intensity measure of the PPP initialization?

Idea:
Choose a target invariant measure:

$$\eta(\mathrm{d}x) = \frac{1}{\exp(x)}\,\mathrm{d}x$$

Use SDE theory:

- Choose $d(x) = \; - 1/2$

Equation for invariant measure

$$0 = \qquad \eta(\mathrm{d}x) - \eta'(\mathrm{d}x)$$

Resulting SDE:

$$\mathrm{d}X_t = -\frac{1}{2}\left(1 \qquad\qquad \right)\mathrm{d}t + \mathrm{d}B_t$$

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Use SDE theory:

- Choose $d(x) = \quad \phi_1(x)$

Equation for invariant measure leads to drift equation:

$$0 = \phi_1(x)\,\eta(\mathrm{d}x) - \eta'(\mathrm{d}x)$$
$$= \phi_1(x)\,\frac{1}{\exp(x)+1} + \frac{\exp(x)}{(\exp(x)+1)^2}$$

Solving gives:

$$\phi_1(x) = 1 - \frac{1}{\exp(x)+1}$$

Resulting SDE:

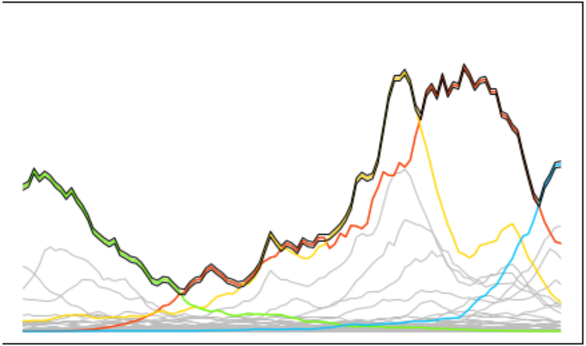
$$\mathrm{d}X_t = -\frac{1}{2}\left(1 - \frac{1}{\exp(X_t)+1}\right)\mathrm{d}t + \mathrm{d}B_t$$

Example — continued

The particle system

$$\left\{ X_t^{(i)} \left(V^{(i)} \right) \mid i \in \mathbb{N} \right\} \qquad \text{is stationary and so is} \qquad \max_{i \in \mathbb{N}} X_t^{(i)} \left(V^{(i)} \right)$$

Note: This is only max-id in general, not max-stable.



Max-domain of attraction

3

Max domain of attraction

For marginal distribution:
A distribution function F is in the max-domain of attraction of an extreme value distribution G (for example $G = \Lambda$ the Gumbel distribution)—in short $F \in \mathrm{MDA}(G)$—if there exist $a_n > 0, b_n \in \mathbb{R}$ such that

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For processes: A process (Y_t) is in the max-domain of attraction of a process (Z_t) if there exist $a_n > 0, b_n \in \mathbb{R}$ such that

$$\max_{i=1,\dots,n} \frac{(Y_t)^{(i)} - b_n}{a_n} \;\to_d\; (Z_t)\,,$$

for example in finite dimensional distributions or in the space of cadlag functions.

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- There exists $a_n > 0, b_n \in \mathbb{R}$ such that $(X_t - b_n)/a_n \to B_t - t/2$ in distribution
- $\max_{i\in\mathbb{N}} X_t^{(i)}(V^{(i)})$ is in the MDA of $\max_{i\in\mathbb{N}}(U^{(i)} + B_t^{(i)} - t/2)$

Revisit Example—SDE perspective

Original SDE:

$$\mathrm{d}X_t \;=\; -\frac{1}{2}\left(1 - \frac{1}{\exp(X_t)+1}\right)\mathrm{d}t \;+\; \mathrm{d}B_t$$

Scaled SDE: Write $X_t^n = X_t - \log(n) \rightsquigarrow a_n = 1$ and $b_n = \log(n)$

$$\mathrm{d}X_t^n \;=\; -\frac{1}{2}\left(1 - \frac{1}{\exp(X_t^n + \log(n))+1}\right)\mathrm{d}t \;+\; \mathrm{d}B_t$$

Limit SDE: Take $n \to \infty$ to get

$$\mathrm{d}X_t^\infty \;=\; -\frac{1}{2}\,\mathrm{d}t \;+\; \mathrm{d}B_t$$

$\rightsquigarrow$ spectral process in the limit: $X_t^\infty = B_t - t/2$ Brownian motion with drift

Revisit Example—Invariant measure perspective

Original:

$$-\log(F(x)) \;=\; \eta([x,\infty)) \;=\; \int_x^\infty \frac{1}{\exp(s)+1}\,\mathrm{d}s$$

Scaled:

$$-\log(F^n(x-\log(n))) \;=\; n\eta([x-\log(n),\infty)) \;=\; \int_x^\infty \frac{1}{\exp(s)+1/n}\,\mathrm{d}s$$

Limit:

$$-\log(\Lambda(x)) \;=\; \int_x^\infty \exp(-s)\,\mathrm{d}s \;=\; \exp(-x)$$

$\rightsquigarrow$ marginal distribution in the limit: Gumbel distribution $\Lambda(x) = \exp(-\exp(-x))$
Takeaway: Our example is in the MDA of the ”classical” Brown-Resnick process

Summary

Motivation:
Understand and extend structure of classical Brown-Resnick process
Construction:
Find invariant process-measure pairs

- stationarity by Markovian particle system started in invariant measure
- find invariant process-measure pair via SDE theory

Max-domain of attraction:
Constructed max-id processes are in the MDA of the classical Brown-Resnick process

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References

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